Washing Theory for Nonequilibrium Stages

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A new approach for the determination of concentrations in the case of nonequilibrium washing stages is presented. Each washing stage is assumed to consist of series of mixing cells, and the conservation of solute mass is solved by Laplace transformation. The simplicity of the resulting equations makes this approach more practical than the existing one.

Introduction

Filtration washing, in general, can be performed in two different ways: displacement washing and reslurry washing. Displacement washing is carried out by passing wash liquor through the filter cake. This method cannot be used when cracks occur in the cake or the resistance to fluid flow through the cake is high. In this case, reslurry washing is preferred where the filter cake and wash liquor are mixed to form a slurry, which is then filtered. Both processes are repeated, crosswise or countercurrentwise, until the desired concentration of solute in the filter cake is achieved. The use of countercurrent stages is usually preferred since they are much more efficient, leading to less wash-liquor consumption.

Countercurrent filtration washing has been extensively used in the industry over the last two decades. Since a belt filter is the major equipment involved in filtration washing, theories have been developed mainly for belt-filter-like washing systems. In the case of equilibrium stages, that is, where filter-cake-liquor holdup and washings leaving any stage have the same solute concentration, the theory is straightforward. However, the washing theory for nonequilibrium stages becomes rather complex.

A displacement washing theory for nonequilibrium stages has been developed by Tomiak (1979, 1982a,b, 1984) using the concept of fractional holdup and fractional displacement. The main disadvantage of this theory, as stated by Tomiak (1984), is the difficulty in understanding the relationship between the characteristic washing curve data and the mathematical development. A somewhat similar procedure is presented by Norden et al. (1982), in which a mass-transfer model is used together with the superposition principle.

In this work an alternative displacement washing theory for nonequilibrium stages is presented. Each washing stage is assumed to consist of a number of mixing cells. The application of the conservation of solute mass together with the macroscopic material balance around each stage yield a set of algebraic equations that can be solved in a straightforward manner. The solution of these equations does not require fractional holdup values to be known, making the proposed approach closer to reality from the engineering point of view. Although the theory is presented for belt filters, it can easily be extended to rotary drum and pan filters as well as dynamic filters, that is, rotary filter press.

Theory

Tomiak's approach

Tomiak stated that for any washing stage shown in Figure 1, the solute concentration in the residual liquor holdup, c, arises as a result of the mixing of the original liquor in the cake and the wash liquor such that

$$c = hc_0 + dw_0, \tag{1}$$

where h is the fractional holdup of the original liquor in the cake and d is the fractional displacement by the wash liquor.

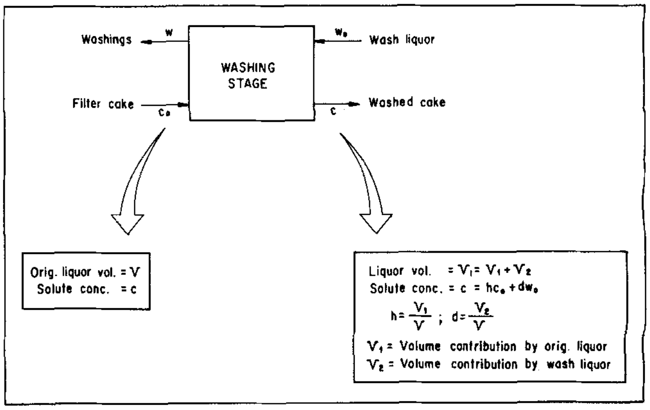

Figure 1. Typical washing stage.

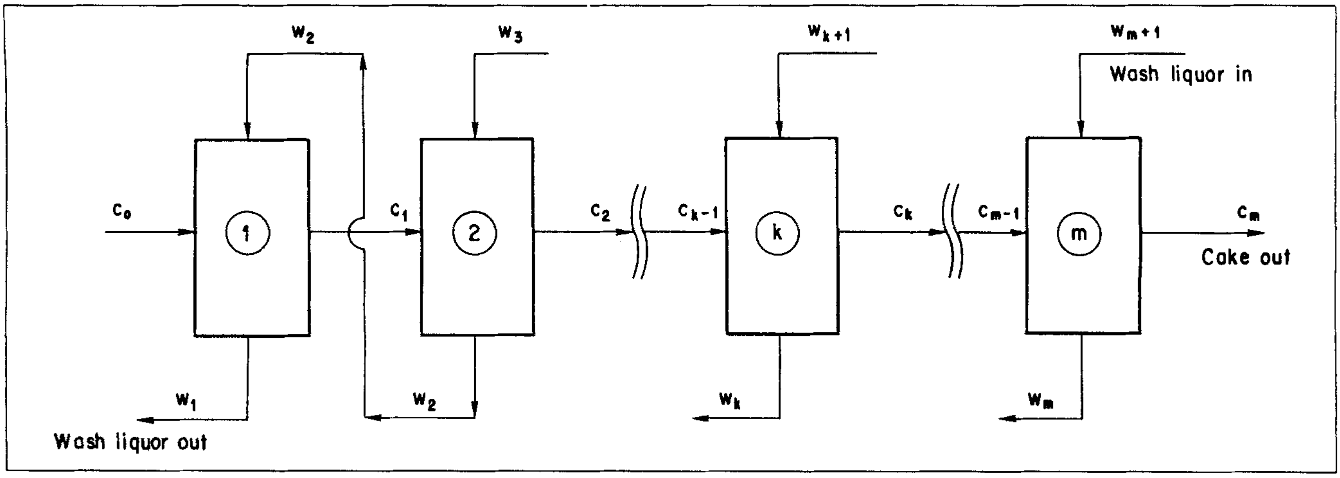

Figure 2. Countercurrent washing stages.

These two quantities are related by

$$h + d = 1. \tag{2}$$

Combination of Eqs. 1 and 2 expresses h and d in terms of concentrations as

$$h = \frac{c - w_0}{c_0 - w_0} \tag{3}$$

$$d = \frac{c_0 - c}{c_0 - w_0}. \tag{4}$$

Hermia and Letesson (1982) experimentally verified that the quantities h and d are independent of concentration.

When the concept of fractional holdup and fractional displacement is extended to multistage washing shown in Figure 2, the average concentration of the liquor holdup in the cake leaving the kth stage is given by

$$c_k = h_k c_0 + \sum_{r=1}^{k} d_{rk} w_{r+1} \qquad \text{for} \quad k = 1,2,\dots,m, \tag{5}$$

with the constraint that

$$h_k + \sum_{r=1}^{k} d_{rk} = 1 \qquad \text{for} \quad k = 1,2,\dots,m. \tag{6}$$

Equation 5 takes the contribution of each wash liquor addition to the liquor concentration into account in subsequent stages. Here d_{rk} represents the fractional displacement in stage k due to wash-liquor addition at stage r. On the other hand, h_k is the fractional holdup of the original liquor in the filter cake leaving the kth stage.

For equal stages, fractional displacements are assumed to be independent of the stage location, but dependent on the relative position of the wash-liquor addition stage. This implies that

$$d_{(k-\sigma)k} = d_{\sigma+1}, \tag{7}$$

so that Eq. 5 reduces to

$$c_k = h_k c_0 + \sum_{\sigma=0}^{k-1} d_{\sigma+1} w_{k-\sigma+1} \qquad \text{for} \quad k = 1,2,\dots,m, \tag{8}$$

with the constraint that

$$h_k + \sum_{\sigma=0}^{k-1} d_{\sigma+1} = 1 \qquad \text{for} \quad k = 1,2,\dots,m. \tag{9}$$

The use of Eq. 9 expresses fractional displacements in terms of fractional holdups in the form

$$d_k = h_{k-1} - h_k, \tag{10}$$

with $h_0 = 1.0$. Substituting Eq. 10 into Eq. 8 gives

$$c_k = h_k c_0 + \sum_{\sigma=0}^{k-1} (h_\sigma - h_{\sigma+1}) w_{k-\sigma+1} \qquad \text{for} \quad k = 1,2,\dots,m. \tag{11}$$

To complete the mathematical picture, the overall material balance around the kth stage is written as

$$c_{k-1} - c_k = N(w_k - w_{k+1}) \qquad \text{for} \quad k = 1,2,\dots,m, \tag{12}$$

where the volumetric wash ratio, N, is the ratio of the wash-liquor volume to the liquid volume in the cake,

$$N = \frac{V_w}{V_L}. \tag{13}$$

The use of Eqs. 8 and 12 to calculate c_k and w_k from the given c_0, w_{m+1}, and N, requires the values of h_k to be known. One way of determining h_k values is to carry out a simple washing test with solute-free wash liquor. In order to get a theoretical expression for h_k, Tomiak (1973) used the perfect-mixing cells in series model and obtained

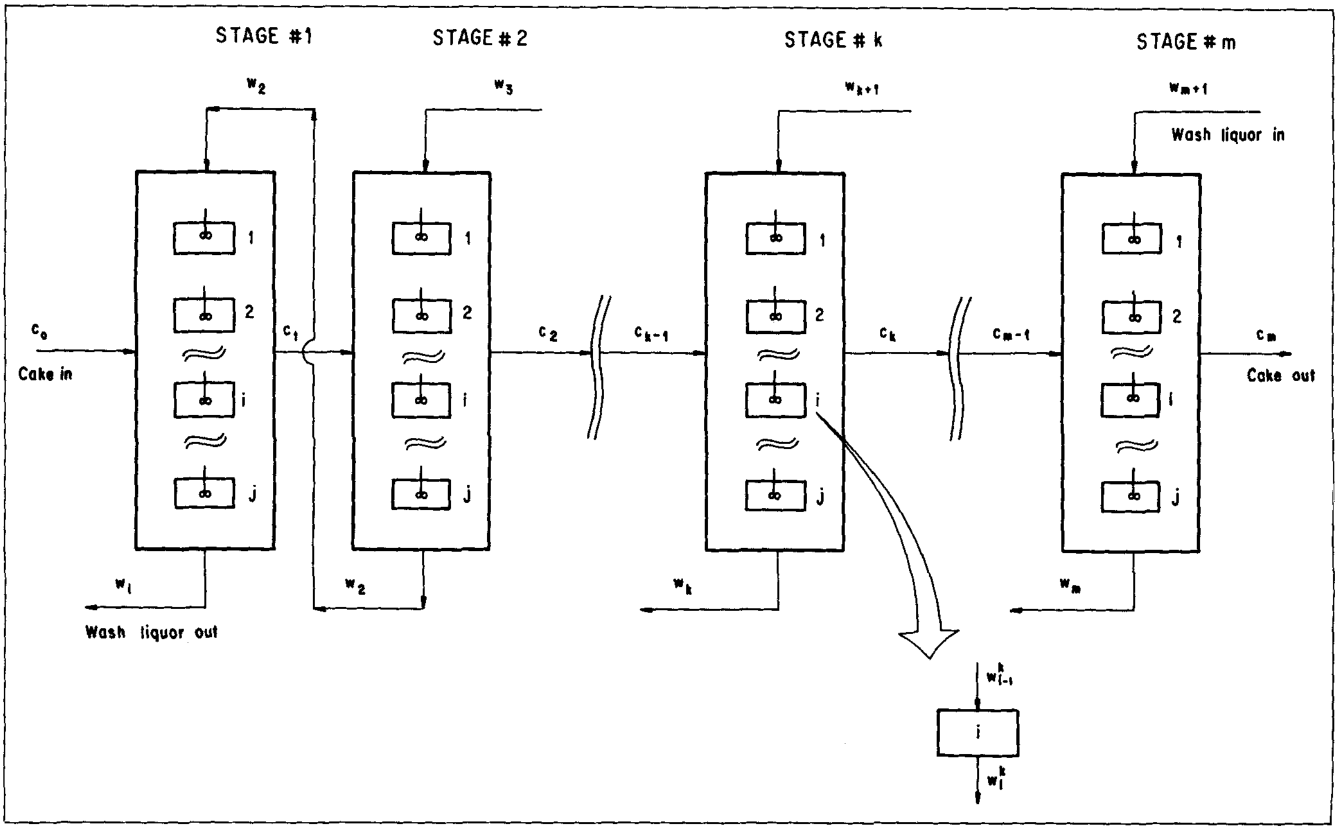

Figure 3. Countercurrent washing stages with mixing cells.

$$h_1(N) = \frac{e^{-jN}}{j} \sum_{\beta=1}^{j} \beta \frac{(jN)^{j-\beta}}{(j-\beta)!}. \tag{14}$$

Without any justification, he further stated that

$$h_k = h_1(kN). \tag{15}$$

Alternative approach

The general case of countercurrent washing with m stages, each containing j mixing cells, is shown in Figure 3. The following assumptions are made:

1. Volumetric liquor holdup of the cake is constant.
2. Each mixing cell holds an equal volume of liquid.
3. The total volume of each mixing cell is constant: inlet and outlet wash-liquor volumetric flow rates are the same.
4. Mixing cell contents are well-mixed so that the concentration of the exit stream is the same as that of the mixing cell contents.
5. Solute is not adsorbed by the cake solids.

Figure 3 also shows the ith mixing cell in the kth stage. The rate equation for the conservation of solute mass in the form

$$\begin{bmatrix}\text{Rate of}\\ \text{solute in}\end{bmatrix} - \begin{bmatrix}\text{Rate of}\\ \text{solute out}\end{bmatrix} = \begin{bmatrix}\text{Rate of solute}\\ \text{accumulation}\end{bmatrix}$$

can be written for the ith mixing cell in the kth stage as

$$\dot{Q}_w w^k_{i-1} - \dot{Q}_w w^k_i = \frac{d}{dt}\left(\frac{V_L}{j} w^k_i\right). \tag{16}$$

Expressing the volumetric wash ratio, N, and the dimensionless time, τ, as

$$N = \frac{t\dot{Q}_w}{V_L} \tag{17}$$

and

$$\tau = jN = \frac{jt\dot{Q}_w}{V_L} \tag{18}$$

reduces Eq. 16 to the form

$$w^k_{i-1} - w^k_i = \frac{dw^k_i}{d\tau}. \tag{19}$$

The initial condition required to solve Eq. 19 is

$$w^k_i = c_{k-1} \quad \text{at} \quad \tau = 0. \tag{20}$$

Taking the Laplace transform of Eq. 19 and using the initial condition gives

$$\bar{w}^k_i(1+s) - \bar{w}^k_{i-1} = c_{k-1}, \tag{21}$$

where the Laplace transform of w_i^k, $\bar{w}_i^k$, is defined by

$$\bar{w}_i^k = \int_0^\infty e^{-s\tau} w_i^k \, d\tau. \tag{22}$$

Equation 21 is a linear difference equation and the solution is given by

$$\bar{w}_i^k = \frac{w_{k+1} + c_{k-1}[(s+1)^i - 1]}{s(s+1)^i}. \tag{23}$$

The details of the solution are given in the Appendix.

Using the Heaviside partial fractions expansion theorem, the inverse Laplace transform of Eq. 23 is

$$w_i^k = w_{k+1} + (c_{k-1} - w_{k+1}) e^{-\tau} \sum_{\ell=1}^{i} \frac{\tau^{\ell-1}}{(\ell-1)!}. \tag{24}$$

Since each stage contains j mixing cells, for $i = j$, Eq. 24 becomes

$$w_j^k = w_{k+1} + (c_{k-1} - w_{k+1}) e^{-\tau} \sum_{\ell=1}^{i} \frac{\tau^{\ell-1}}{(\ell-1)!}. \tag{25}$$

The concentration of the wash liquor exiting the kth stage is given by

$$w_k = \frac{1}{\tau} \int_0^\tau w_j^k \, d\tau. \tag{26}$$

Substitution of Eq. 25 into Eq. 26 gives

$$w_k = (1 - I) w_{k+1} + I c_{k-1} \qquad \text{for } k = 1, 2, \ldots, m, \tag{27}$$

where

$$I = \frac{1}{N} - e^{-\tau} \sum_{\beta=1}^{j} \beta \frac{\tau^{j-\beta-1}}{(j-\beta)!} = \frac{1-h_1}{N}. \tag{28}$$

For the given values of c_0, w_{m+1}, N, and j, the concentrations c_k and w_k are calculated from Eqs. 12 and 27.

Determination of the number of mixing cells

The use of Eqs. 12 and 27 requires the number of mixing cells, j, to be known. This can be determined as follows.

For a single-stage washing process with a solute-free wash liquor, Eqs. 12 and 27 reduce to

$$c_0 - c_1 = N w_1 \tag{29}$$

and

$$w_1 = I c_0. \tag{30}$$

Elimination of w_1 between Eqs. 29 and 30 gives

$$\frac{c_1}{c_0} = 1 - NI. \tag{31}$$

Substitution of Eq. 28 into Eq. 31, and making use of Eq. 18 yields

$$\frac{c_1}{c_0} = \frac{e^{-jN}}{j} \sum_{\beta=1}^{j} \beta \frac{(jN)^{j-\beta}}{(j-\beta)!} \tag{32}$$

In Figure 4, the ratio c_1/c_0 is plotted vs. volumetric wash ratio N with j as a parameter. Therefore, if one performs a washing experiment with solute-free wash liquor with a known value of volumetric wash ratio N and measures c_1/c_0, then the value of j can be determined from Figure 4.

Example

Let us consider the washing of a filter cake using 5 countercurrent stages with a solute-free wash liquor, i.e., $w_6 = 0$, when the volumetric wash ratio is 1.1. This example was presented by Tomiak (1979, 1984) in several places. For $j = 2$, i.e., two mixing cells in each stage, Eq. 28 gives

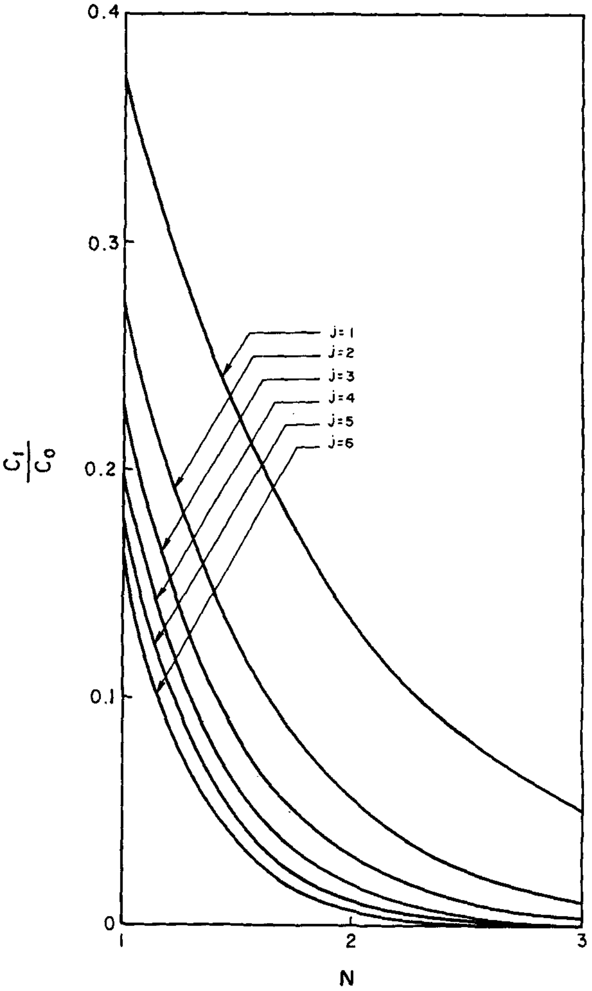

Figure 4. Computed values of c_1/c_0 vs. N with j as a parameter.

Table 1. Cake Liquor and Washings Concentrations for Countercurrent Washing Process with Five Stages*

Conc.	This Work	Tomiak
c_1	$0.6945c_0$	$0.7042c_0$
c_2	$0.4595c_0$	$0.4577c_0$
c_3	$0.2786c_0$	$0.2718c_0$
c_4	$0.1395c_0$	$0.1325c_0$
c_5	$0.0325c_0$	$0.0282c_0$
w_1	$0.8796c_0$	$0.8834c_0$
w_2	$0.6019c_0$	$0.6145c_0$
w_3	$0.3882c_0$	$0.3904c_0$
w_4	$0.2238c_0$	$0.2214c_0$
w_5	$0.0973c_0$	$0.0948c_0$

*Each containing two mixing cells, when the volumetric wash ratio is 1.1.

$$I=\frac{1}{1.1}-e^{-2.2}\sum_{\beta=1}^{2}\beta\frac{(2.2)^{1-\beta}}{(2-\beta)!}=0.6976$$

For each stage, Eq. 27 is written as

$$\begin{aligned}w_1&=0.3024w_2+0.6976c_0\\ w_2&=0.3024w_3+0.6976c_1\\ w_3&=0.3024w_4+0.6976c_2\\ w_4&=0.3024w_5+0.6976c_3\\ w_5&=0.6976c_4\end{aligned}$$

On the other hand, for each stage Eq. 12 is written as

$$\begin{aligned}c_0-c_1&=1.1(w_1-w_2)\\ c_1-c_2&=1.1(w_2-w_3)\\ c_2-c_3&=1.1(w_3-w_4)\\ c_3-c_4&=1.1(w_4-w_5)\\ c_4-c_5&=1.1w_5.\end{aligned}$$

The preceding equations can be solved simultaneously in a straightforward manner. The results are given as multiples of c_0 in Table 1.

Note that there is a small discrepancy between the predictions of this work and those of Tomiak. This is due to the calculations of h_k values in Tomiak's approach, as will be explained in the next section.

Comparison of Approaches

In a countercurrent washing operation with m stages and known values of volumetric wash ratio, N, original cake-liquor and wash-liquor concentrations, c_0 and w_{m+1}, the number of unknown concentrations is $2m$, that is, $c_1, c_2, \ldots, c_m$ and $w_1, w_2, \ldots, w_m$. The macroscopic solute material balance around each washing stage, Eq. 12, provides m equations. In order to get a determinate mathematical system, m, more equations are needed.

In Tomiak's approach, the concentration of cake liquor leaving the kth stage is expressed as fractions of the original cake-liquor concentration, h_k, and wash-liquor concentrations prior to the kth stage, $d_1, d_2, \ldots, d_k$. To calculate these fractions, a *single* washing stage is assumed to consist of a series of mixing cells, and a theoretical expression for h_1 is obtained as a function of volumetric wash ratio. Then Eq. 15 is proposed to calculate the values of h_k.

The approach presented in this article, on the other hand, does not require h_k values to be calculated. Assuming *each* washing stage to consist of a series of mixing cells, the solution of the conservation of solute mass by Laplace transformation yields Eq. 27, which provides m more equations. The relationship between the two approaches can be obtained as follows.

Elimination of w_k between Eqs. 12 and 27, and the use of Eq. 28 expresses c_k in the form

$$c_k=h_1c_{k-1}+(1-h_1)w_{k+1}. \tag{33}$$

Equation 33 leads to the following set of equations:

$$c_1=h_1c_0+(1-h_1)w_2 \tag{34}$$

$$\begin{aligned}c_2&=h_1c_1+(1-h_1)w_3\\ &=h_1^2c_0+(h_1-h_1^2)w_2+(1-h_1)w_3\end{aligned} \tag{35}$$

$$\begin{aligned}c_3&=h_1c_2+(1-h_1)w_4\\ &=h_1^3c_0+(h_1^2-h_1^3)w_2+(h_1-h_1^2)w_3+(1-h_1)w_4\end{aligned} \tag{36}$$

$$\vdots$$

$$c_k=h_1^kc_0+\sum_{\sigma=0}^{k-1}(h_1^\sigma-h_1^{\sigma+1})w_{k-\sigma+1}. \tag{37}$$

Comparison of Eq. 37 with Eq. 11 indicates that

$$h_k=h_1^k, \tag{38}$$

which is different from the expression proposed by Tomiak (Eq. 15).

Conclusions

A displacement washing theory for nonequilibrium stages is presented. Each washing stage is assumed to consist of a series of mixing cells and the conservation of solute mass is solved by the method of Laplace transformation. The resulting equations require the number of mixing cells in each stage to be known. This can be determined by carrying out a simple washing experiment in a laboratory with solute-free wash liquor and a specified value of volumetric wash ratio.

Acknowledgments

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Notation

h_1 = value of h for the first stage
$\dot{Q}_w$ = volumetric flow rate of wash liquor, m^3/s
s = Laplace transform variable
t = time, s
V_L = volume of liquid in the filter cake, m^3
V_w = volume of wash liquor, m^3
w = solute concentration in the wash liquor, kg/m^3

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Appendix

Equation 21 is a linear difference equation, and the solution is the sum of the homogeneous and particular solutions. To obtain the solution to the homogeneous part,

$$\bar{w}_i^k(1+s) - \bar{w}_{i-1}^k = 0, \tag{A1}$$

substitution of the trial function

$$\bar{w}_i^k = \Omega^i \tag{A2}$$

into Eq. A1 gives

$$\Omega = \frac{1}{1+s}. \tag{A3}$$

Hence, the homogeneous solution is

$$\bar{w}_i^k = \frac{A}{(1+s)^i}, \tag{A4}$$

where A is a constant. On the other hand, the particular solution can be easily determined as

$$\bar{w}_i^k = \frac{c_{k-1}}{s}. \tag{A5}$$

The sum of Eqs. A4 and A5 gives the total solution as

$$\bar{w}_i^k = \frac{A}{(1+s)^i} + \frac{c_{k-1}}{s}. \tag{A6}$$

When $i = 0$, Eq. A6 reduces to

$$\bar{w}_0^k = A + \frac{c_{k-1}}{s}. \tag{A7}$$

On the other hand, the use of Eq. 26 gives

$$\bar{w}_0^k = \int_0^\infty e^{-s\tau} w_0^k \, d\tau = \frac{w_{k+1}}{s}. \tag{A8}$$

Comparison of Eqs. A7 and A8 gives the value of the constant A as

$$A = \frac{w_{k+1} - c_{k-1}}{s}. \tag{A9}$$

Substitution of Eq. A9 into Eq. A6 and rearrangement gives Eq. 23.

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